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STOCHASTIC STABILITY OF FORCED NONLINEAR SHELL VIBRATIONS

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A. S. VOL'MIR and Kh. P. KUL'TERBAEV
(Moscow)

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The parameters of stationary forced nonlinear vibrations are determined within the framework of correlation theory for shells considered as a system with one degree of freedom and subjected to a transverse pressure which is random in time. The generalized force is described as a stationary normal process with a rational fraction spectral density.

The stability of the solutions found is verified by the perturbed motion equation in a linear approximation. The system is first reduced to a Markov type by extension of the phase space. Then the Liapunov theorem on stability in a linear approximation is applied to the set of first and second order moment functions. The final stage in the problem is executed by numerical methods.

It is disclosed that there are unstable solutions in some domain of the parameter space. Jump-like transitions from some stable states to others are observed for systems with comparatively large nonlinearity.

Characteristic kinds of deterministic loadings have been investigated in [1–4]. For essentially nonlinear systems the curves of the states have sections corresponding to unstable motions.

Stationary forced vibrations of shells under random loads have been examined in a number of papers [5–7]. Investigations conducted within the framework of the correlation approximation often yield ambiguous solutions and the question of what motions are realized, remains open.

The main purpose herein is to extract those of the solutions which correspond to the unstable vibrations, and thereby determine the actual shell behavior more accurately.

1. The behavior of a system with one degree of freedom in the nonlinear dynamics of elastic plates and shells is often described by a differential equation of the following kind [1]:

$$y^{\cdot\cdot} + 2\varepsilon y^{\cdot} + y - \alpha y^2 + \beta y^3 = q \tag{1.1}$$

Here $y(\tau)$ is the generalized coordinate, ε is the damping coefficient, α, β are constants, and $q(\tau)$ is a generalized force. Differentiation is performed with respect to the dimensionless time $\tau = \omega_0 t$, where ω_0 is the circular frequency of small natural vibrations and t is the time.

Let us find the mathematical expectation m_y and the variance σ_y^2 of the output process if $q(\tau)$ is a stationary normal process with the mathematical expectation m_q, variance σ_q^2 and spectral density (θ is a positive nonrandom constant)

$$S_q(\omega) = \frac{\sigma_q^2\theta}{\pi}\frac{1}{\theta^2 + \omega^2} \tag{1.2}$$

We use the method of spectral representations [8] in combination with the "quasi-Gauss" hypothesis [9] which assumes the same relationships for the moment functions of the output process as for the normal process. Let us introduce the Fourier-Stieltjes expansions

$$y(\tau) = m_y + \int_{-\infty}^{\infty} Y(\omega)\, e^{i\omega\tau} d\omega \tag{1.3}$$

$$q(\tau) = m_q + \int_{-\infty}^{\infty} Q(\omega)\, e^{i\omega\tau} d\omega$$

where the spectra $V(\omega)$ and $Q(\omega)$ possess the stochastic orthogonality property

$$\langle Y(\omega) Y^*(\omega')\rangle = S_y(\omega)\delta(\omega - \omega')$$
$$\langle Q(\omega) Q^*(\omega')\rangle = S_q(\omega)\delta(\omega - \omega')$$

Here and henceforth, the angular brackets denote the operation of taking the average over the set of realizations, the asterisks denote the passage to the complex-conjugate, $\delta(\omega)$ is the Dirac delta function, and $S_y(\omega)$ is the spectral density of the generalized coordinate. Substituting (1.3) into (1.1), we obtain an expression of the type

$$f(Y, Q, m_y, m_q, \ldots) = 0 \tag{1.4}$$

We take the average of this latter taking into account that the process $y(\tau)$ is quasi-Gaussian and stationary, and we arrive at the dependence

$$m_y - \alpha m_y^2 + \beta m_y^3 + (3\beta m_y - \alpha)\sigma_y^2 - m_q = 0 \tag{1.5}$$

Applying the convolution operation for random spectra [10] to (1.4), we write

$$\begin{aligned}&[(i\omega)^2 + 2\varepsilon(i\omega) + 1 - 2\alpha m_y + 3\beta m_y^2]\, Y(\omega) + \\ &(m_y - \alpha m_y^2 + \beta m_y^3 - m_q)\,\delta(\omega) + \\ &(3\beta m_y - \alpha)\int_{-\infty}^{\infty} Y(\omega_1)\, Y(\omega - \omega_1)\, d\omega_1 + \\ &\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(\omega_1)\, Y(\omega_2)\, Y(\omega - \omega_1 - \omega_2)\, d\omega_1 d\omega_2 - Q(\omega) = 0\end{aligned} \tag{1.6}$$

Let us examine (1.6) and its complex-conjugate equation. Multiplying their left sides by $Y^* (\omega')$ and $Q (\omega')$, respectively, and averaging the results we obtain

$$L_y (i\omega)S_y (\omega) - S_{qy} (\omega) = 0, \qquad L_y (-i\omega)S_{qy} (\omega) - S_q (\omega) = 0 \qquad (1.7)$$
$$L_y (i\omega) = (i\omega)^2 + 2\varepsilon (i\omega) + g, \quad g = 1 - 2\alpha m_y + 3\beta (m_y^2 + \sigma_y^2)$$

Here $S_{qy} (\omega)$ is the mutual spectral density of the processes $q (\tau)$ and $y (\tau)$. Taking into account (1.2), we have from the system (1.7)

$$S_y(\omega) = \frac{\sigma_q^2\theta}{\pi} \frac{1}{L (i\omega) L (-i\omega)} \qquad (1.8)$$
$$L (i\omega) = (i\omega)^3 + \mu (i\omega)^2 + \nu (i\omega) + g\theta$$
$$\mu = 2\varepsilon + \theta, \qquad \nu = g + 2\varepsilon\theta$$

Integrating (1.8) with respect to ω we obtain the second relation between m_y and σ_y^2

$$2\varepsilon\sigma_y^2 g (\mu\theta + g) - \mu\sigma_q^2 = 0 \qquad (1.9)$$

Equations (1.5) and (1.9) form a nonlinear algebraic system in the unknowns m_y and σ_y^2, which contains high degree terms; explicit expressions cannot successfully be written down for them. An analysis of the results by using a digital computer disclosed that the solution is multivalued for certain relationships between the system parameters. It is natural to assume that some of them correspond to unstable motions; the remainder of the paper is given over to determining them.

2. Let us form the perturbed motion equation in a linear approximation

$$x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + (\gamma + \zeta y_0 + 3\beta y_0^2) x = 0 \qquad (2.1)$$
$$\gamma = 1 - 2\alpha m_y + 3\beta m_y^2, \qquad \zeta = 2 (3\beta m_y - \alpha)$$

Here $x (\tau)$ is the variation of the generalized coordinate which has stochastic meaning; the random process $y (\tau)$ is represented as the sum of the mathematical expectation m_y and the centralized component $y_0(\tau)$: $y (\tau) = m_y + y_0 (\tau)$.

The expression (2.1) is the stochastic analog of the Mathieu-Hill equations in which the stationary random process $y_0 (\tau)$ with the spectral density (1.8) plays the part of the excitation function or the external parametric effect. In this case the problem under consideration has much in common with the parametric resonances in stochastic systems [9].

The stochastic stability will be identified with Liapunov stability of the set of mathematical expectations and the second order moment functions for the components of a Markov process. In order for the behavior of the system to be considered a Markov process, it should be described by first order differential equations and the effect should be delta-correlated. The expression (2.1) does not satisfy these conditions, but the perturbed motion process can be reduced to Markov type by the introduction of a sufficiently broad set of "coordinates".

The second condition can be satisfied by considering $y_0 (\tau)$ as the result of the passage of white noise $z (\tau)$ with intensity $s = 2\sigma_q^2\theta$ through a filter

$$y_0^{\cdot\cdot\cdot} + \mu y_0^{\cdot\cdot} + \nu y_0^{\cdot} + g\theta y_0 = z \qquad (2.2$$

The first condition is satisfied if we go from (2.1) and (2.2) to the following system of first order differential equations

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -2\varepsilon x_2 - (\gamma + \zeta x_3 + 3\beta x_3^2)x_1 \tag{2.3}$$
$$x_3^{\cdot} = x_4, \quad x_4^{\cdot} = x_5, \quad x_5^{\cdot} = -\mu x_5 - \nu x_4 - g\theta x_3 + z$$

We then have the vector Markov process $\mathbf{x}(\tau)$ with the components

$$x_1 = x, \ x_2 = x^{\cdot}, \ x_3 = y_0, \ x_4 = y_0^{\cdot}, \ x_5 = y_0^{\cdot\cdot}$$

Furthermore, the stochastic stability is verified by the deterministic linearized system

$$\mathbf{m}^{\cdot} = A\mathbf{m} \tag{2.4}$$
$$\mathbf{m} = \{m_1, m_2, m_{11}, m_{12}, m_{13}, m_{14}, m_{15}, m_{22}, m_{23}, m_{24}, m_{25}\}$$

where A is a matrix of constant coefficients, $\mathbf{m}(\tau)$ is a vector of first and second order moment functions with identical components, determined as a result of the following averagings:

$$m_i(\tau) = \langle x_i(\tau)\rangle, \quad m_{ij}(\tau) = \langle x_i(\tau)\ x_j(\tau)\rangle \tag{2.5}$$
$$i = 1, 2; \quad j = 1, 2, \ldots 5$$

The remaining moment functions refer just to the stationary process $y_0(\tau)$; hence they take no part in the definition of stability. Let us write those down which are needed for the subsequent computations

$$m_3 = m_4 = m_5 = m_{34} = 0, \qquad m_{33} = \sigma_y^2$$
$$m_{35} = \frac{\sigma_q^2\theta}{g\theta - \mu\nu}$$

A part of them is calculated by using (1.8). Using (2.3), (2.5) and the commutativity of the differentiation and averaging operations, all the equations in the system (2.4) can be formed

$$m_i^{\cdot} = \langle x_i^{\cdot}\rangle, \quad m_{ij}^{\cdot} = \langle x_i^{\cdot}x_j\rangle + \langle x_i x_j^{\cdot}\rangle \tag{2.6}$$
$$i = 1, 2; \quad j = 1, 2, \ldots 5$$

The third and fourth order moment functions in the right sides of (2.6) are eliminated from the subsequent analysis by using the quasi-Gaussian hypothesis. After discarding the nonlinear terms, the matrix A becomes

$$A = \left\| \begin{array}{ccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g & -2\varepsilon & 0 & 0 & -\zeta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -g & -2\varepsilon & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -g\theta & -\nu & -\mu & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2g & 0 & 0 & 0 & -4\varepsilon & 0 & 0 & 0 \\
-\zeta\sigma_y^2 & 0 & 0 & 0 & -g_1 & 0 & 0 & 0 & -2\varepsilon & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -g & 0 & 0 & 0 & -2\varepsilon & 1 \\
-\eta & 0 & 0 & 0 & \xi & 0 & -g & 0 & -g\theta & -\nu & -2\varepsilon-\mu
\end{array} \right\|$$

$(\eta = \zeta m_{35}, \ \zeta = 6\beta m_{35}, \ g_1 = g + 6\beta\sigma_y^2)$

The stability analysis henceforth reduces to seeking the roots of the characteristic equation

$$\det (A - \lambda E) = 0 \qquad (2.7)$$

where λ is the characteristic exponent, and E is the unit matrix. Because of the high order of the matrix A, the derivation of analytical expressions is difficult. The results obtained were analyzed by numerical methods on a digital computer.

3. Two cases of nonlinearity coefficients: (1) $\alpha = 0.652$, $\beta = 0.155$ (Figs. 1 and 2) and (2) $\alpha = 0.460$, $\beta = 0.055$ (Figs. 3 and 4) are considered for the following fixed system parameters: $\theta = 0.02$, $\varepsilon = 0.01$ and $\sigma_q / m_q = 0.4$. The second case corresponds to a shell with one unstable and two stable equilibrium positions under static loading.

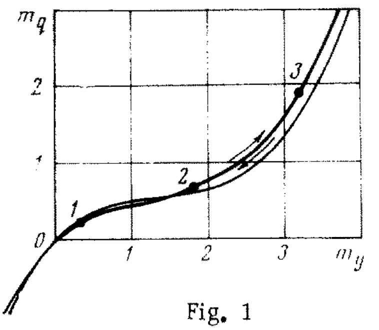

Fig. 1

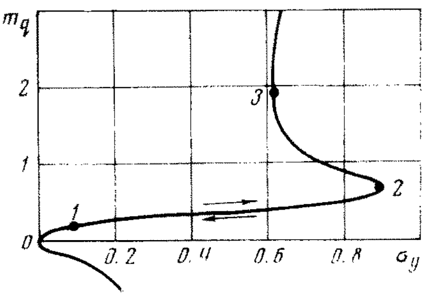

Fig. 2

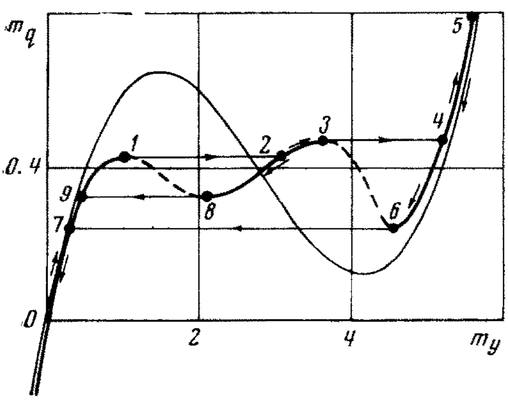

Fig. 3

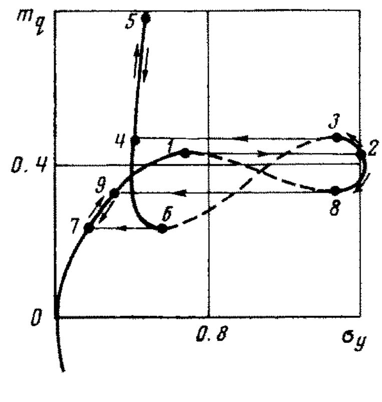

Fig. 4

The graphs of $m_q - m_y$ (Figs. 1 and 3) and $m_q - \sigma_y$ (Figs. 2 and 4) are initially constructed by using (1.5) and (1.9). Curves of the static deflections $q - y$ are superposed in Figs. 1 and 3 by a fine line for comparison. Then a sufficiently large quantity of points confirmed the stability by means of (2.7). In all cases the roots obtained satisfy the conditions of the Liapunov theorem on stability in a first approximation. No instability is detected for the first shell, while for the second one the descending parts of the curves marked by dashes correspond to the unstable motions.

In the first case, the mathematical expectation and the standard deviation vary smoothly as m_q rises and (or) drops. Let us trace the shell behavior as the mean load grows from zero. The initial section 0 - 1, corresponding to comparatively low values of m_q, m_y, σ_y is supplanted by a section of rapid growth of the mathematical expectation and the standard deviation (1 - 2) when the shell reaches the least stiffness. Further growth in the load causes a drop in σ_y and retardation of the increment in m_y (2 - 3). The values of m_y and σ_y grow monotonely after the point 3. As m_q decreases, we have a change of states and the reverse sequence. For negative m_q the values of m_y and σ_y are much smaller because of the high shell stiffness.

The behavior of the system is more complex in the second case. In the initial loading period, vibrations primarily around the first stable equilibrium position (the section 0 - 1) with small values of the mathematical expectation and standard deviation are observed here. At the point 1 this state is replaced by a jump by vibrations enclosing both stable equilibrium positions (the section 8 - 2 - 3). The values of m_y and σ_y rise sharply. A further rise in m_q results in the jump 3 - 4 denoting the passage to vibrations primarily around the snapping state; the standard deviation drops strongly because of the growing stiffness. The decrease in m_q on the sections 3 - 2 - 8 results in the inverse replacement of vibrations enclosing the stable equilibrium positions by vibrations around the first stable equilibrium position (the jump 8 - 9) but this phenomenon occurs for considerably lower values of m_q. An analogous picture is observed as the load decreases on the section 5 - 4 - 6. Here the vibrations around the snapping state are replaced by the jump 6 - 7 by vibrations around the first stable position.

The passages from some stable stationary modes to others noted by arrows in Figs. 3 and 4 are most probable for sufficiently slow changes in m_q. Other variations are not excluded because of the random nature of the load, but they will evidently be less probable.

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THE NONHOLONOMIC PROPERTY OF THE ELASTOPLASTIC STATE OF A MEDIUM AND THE CONDITIONS AT STRONG DISCONTINUITIES

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N. D. VERVEIKO and V. N. NIKOLAEVSKII
(Voronezh, Moscow)

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Theory of discontinuities is used to investigate the conditions at a shock wave in an elastoplastic medium with a nonassociative flow rule. A system of relations is proposed at the shock wave which represents, in general, the nonholonomic conditions which become integrable only when the problem of motion of the medium behind the wavefront is solved. In the present case, the Hugoniot adiabate independent of the flow behind the wavefront is absent.

Equations for determining the plastic deformations of materials are generally written in terms of increments and must be integrated when solving specific problems. If the problems are further complicated by the presence of surfaces of strong discontinuities, then the integration can only be performed when the usual equilibrium relations are supplemented by additional boundary conditions at these surfaces. In the present paper we show that, in the absence of the displacement discontinuities, such a condition must be given in the form of the condition of continuity of displacements. The analysis is carried out with the finite character of the deformations taken into account.

The defining incremental constraints are nonholonomic [1] and cannot, in general, be integrated independently. In such cases the relations connecting the parameters of the system at the strong discontinuities cannot be reduced to a system of finite, closed relations. Thus the Hugoniot adiabate will not, in general, exist in dilating plastic materials [2] irrespective of the motion outside the strong discontinuity.

Some authors [3−5] construct additional relations at the strong discontinuity (they can be used to obtain finite relations across the shock) by analyzing the inner structure of the discontinuity, with the help of the same defining equations sometimes supplemented by viscosity terms and a hypothetical loading route. The specific character of the conditions at the strong discontinuity obtained by the passage to the limit from the continuous structure, was noted by Sedov in [1].

In accordance with the approach developed in this paper, we must consider the structure of the shock transition in order to estimate the changes in the initial state (reference state) of the material point passing across the shock front. For this reason the system of equations for the structure must be chosen, in order to be adequate, from the continuous generalized models.